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The aim of this note is to make sense of [Gro66, Définition 1.1]. The point is that one can define crystals in any category fibered over the category of schemes over a base scheme. This works in any characteristic, the difference is that if the base if of characteristic 0, then once has to replace the crystalline site by the infinitesimal site.

0.1. **Preliminaries on fibered categories.** Recall that if $\mathcal{C}$ is a category and $p : \mathcal{X} \to \mathcal{C}$ a fibered category over $\mathcal{C}$, then $\mathcal{X}$ contains a subcategory $\mathcal{X}_{\mathrm{cart}}$ which consists of all cartesian arrows in $\mathcal{X}$. In fact, the restriction of $p : \mathcal{X} \to \mathcal{C}$ along $\mathcal{X}_{cart} \hookrightarrow \mathcal{X}$ makes $p : \mathcal{X}_{\mathrm{cart}} \to \mathcal{C}$ a category fibered in groupoids over $\mathcal{C}$.

Now suppose $q : \mathcal{Y} \to \mathcal{C}$ and $p : \mathcal{X} \to \mathcal{C}$. A morphism of fibered categories over $\mathcal{C}$ is a morphism $\sigma : \mathcal{Y} \to \mathcal{X}$ so that $q\sigma = p$ and cartesian arrows in $\mathcal{Y}$ are mapped to cartesian arrows in $\mathcal{X}$. Let us denote the category of all such morphisms as $\mathrm{Hom}_{\mathcal{C}}(\mathcal{X}, \mathcal{Y})$. This is a full subcategory of all functors between (the underlying category of) $\mathcal{X}$ and $\mathcal{Y}$.

We want a special case of the above construction. Note that we can transform $\mathcal{C}$ into a fibered category over itself via the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. This makes all arrows of $\mathcal{C}$ cartesian. Let $p : \mathcal{X} \to \mathcal{C}$ be as above. Then $\sigma \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{X})$ satisfy $p \circ \sigma = id_{\mathcal{C}}$ and σ maps all arrows of $\mathcal{C}$ to cartesian arrows in $\mathcal{X}$. Thus the essential image of $\sigma \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{X})$ is always contained in $\mathcal{X}_{\mathrm{cart}}$. Explicitly, this means that for each $c \in \mathcal{C}$, we consider *only* a subcategory of the isomorphisms in the fiber $\mathcal{X}(c)$. Morphisms in $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{X})$ are called *cartesian sections* of $p : \mathcal{X} \to \mathcal{C}$.

A final fact that we will need is the following lemma, whose proof is a definition chase.

Lemma 0.1. *Let $p : \mathcal{X} \to \mathcal{C}$ be a fibered category and let $q : \mathcal{G} \to \mathcal{C}$ be an arbitrary morphism (of categories). Then the base change $p_1 : \mathcal{X} \times_{\mathcal{C}} \mathcal{G} \to \mathcal{G}$ is a fibered category over $\mathcal{G}$*[1].

Proof. As mentioned, this is a definition chase. The point is to observe that if $a \to b$ is an arrow in $\mathcal{G}$ then we can look at $q(a) \to q(b)$ as an arrow in $\mathcal{C}$. Because $\mathcal{X} \to \mathcal{C}$ is fibered, we have a cartesian arrow lifting $q(a) \to q(b)$ in $\mathcal{X}$. Now we can lift this to a cartesian arrow in $\mathcal{X} \times_{\mathcal{C}} \mathcal{G}$ essentially by definition of the fiber product. □

We will now be interested in cartesian sections i.e. objects of the category $\mathrm{Hom}_{\mathcal{G}}(\mathcal{G}, \mathcal{X} \times_{\mathcal{C}} \mathcal{G})$ where we consider $\mathcal{G}$ as fibered over itself with the identity functor. Since a cartesian section $\sigma : \mathcal{G} \to \mathcal{X} \times_{\mathcal{C}} \mathcal{G}$ sits in the diagram below

$$\begin{tikzcd}
\mathcal{G} \arrow[dr, "\sigma"] \arrow[drr, bend left, "id_{\mathcal{G}}"] \arrow[ddr, bend right, "\tau"'] & & \\
& \mathcal{X} \times_{\mathcal{C}} \mathcal{G} \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathcal{G} \arrow[d, "q"] \\
& \mathcal{X} \arrow[r, "p"] & \mathcal{C}
\end{tikzcd} \tag{0.1}$$

the data of σ is completely determined by the map $\tau \colon \mathcal{G} \to \mathcal{X}$. This is the data which assigns to each $a \in \mathcal{G}$ an object $b \in \mathcal{X}$ so that $p(b) = q(a)$ and so that if $f : a \to a'$ is an arrow in $\mathcal{G}$, then one gets an arrow $b \to b'$ in $\mathcal{X}$. It is clear that the essential image of $\tau \colon \mathcal{G} \to \mathcal{X}$ must be $\mathcal{X}_{\mathrm{cart}}$.

0.2. **The fibered category of quasi-coherent modules.** Fix a base scheme S and let Sch_S denote the category of schemes over S. We will now recall the definition of the fibered category Qcoh over Sch_S. In fact most of what we do here can be done over an arbitrary fibered category $\mathcal{X} \to \mathrm{Sch}_S$ (and hence arbitrary stacks).

The category Qcoh over Sch_S has as objects pairs $(\mathcal{F}, Y)$ where Y is an S scheme and $\mathcal{F} \in \mathrm{Qcoh}(Y)$. Arrows in Qcoh consist of pairs $(f, \varepsilon) \colon (\mathcal{F}, Y) \to (\mathcal{F}', Y')$ where $f : Y \to Y'$ is a morphism over S and $\varepsilon : f^*\mathcal{F}' \to \mathcal{F}$ is a morphism in $\mathrm{Qcoh}(Y)$. The canonical functor $q : \mathrm{Qcoh} \to \mathrm{Sch}_S$, sending $(\mathcal{F}, Y) \mapsto Y$, makes Qcoh a fibered category over Sch_S. The fibre above $Y \in \mathrm{Sch}_S$ consists of $\mathrm{Qcoh}(Y)$.

[1]Here the base change is the same as the fibre product in the 2-category of categories over $\mathcal{C}$. See Tag 02XG.

Now we can consider Sch_S as trivially fibered over itself via the identity functor $\mathrm{id}_{\mathrm{Sch}_S}\colon \mathrm{Sch}_S \to \mathrm{Sch}_S$. We can now describe the a cartesian section $\sigma \in \mathrm{Hom}_{\mathrm{Sch}_S}(\mathrm{Sch}_S, \mathrm{Qcoh})$ explicitly as follows–for each $Y \in \mathrm{Sch}_S$ we assign a pair $(\mathcal{F}, Y)$ and for each arrow $g\colon Y \to Y'$ a pair $(g, \theta) : (\mathcal{F}, Y) \to (\mathcal{F}', Y')$ where $g : Y \to Y'$ is the obvious map and $\theta\colon g^*\mathcal{F}' \to \mathcal{F}$ is an *isomorphism* in $\mathrm{Qcoh}(Y)$.

0.3. **Crystals.** We are now ready to describe crystals in quasi-coherent modules (and more generally fibered categories). Let S and Sch_S be as above, and assume that S now comes with a divided power structure $(S, \mathcal{I}, \gamma)$ where $\mathcal{I} \subset \mathcal{O}_S$ is a quasi-coherent ideal and $\gamma : \mathcal{I} \to \mathcal{I}$ is a PD-structure. Let $(X, \mathcal{J}, \eta)$ be a PD-scheme over $(S, \mathcal{I}, \gamma)$ i.e. the strutucure morphism is compatible with the PD-structures. Let $\mathrm{Cris}(X/S)$ denote the crystalline site of X/S.

There is a canonical functor $p : \mathrm{Cris}(X/S) \to \mathrm{Sch}_S$ which maps $(U \hookrightarrow T, \gamma) \to T/S$. We now get a fibre diagram

$$\begin{tikzcd} \mathrm{Qcoh} \times_{\mathrm{Sch}_S} \mathrm{Cris}(X/S) \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathrm{Cris}(X/S) \arrow[d, "p"] \\ \mathrm{Qcoh} \arrow[r, "q"'] & \mathrm{Sch}_S \end{tikzcd}$$

which makes $\mathrm{Qcoh} \times_{\mathrm{Sch}_S} \mathrm{Cris}(X/S) \xrightarrow{p_1} \mathrm{Cris}(X/S)$ a fibered category over $\mathrm{Cris}(X/S)$ by virtue of Lemma (0.1). Explicitly, $\mathrm{Qcoh} \times_{\mathrm{Sch}_S} \mathrm{Cris}(X/S)$ consists of triples $((\mathcal{F}, T'), (U \hookrightarrow T, \gamma), \theta)$ where $(\mathcal{F}, T') \in \mathrm{Qcoh}$, $(U \hookrightarrow T, \gamma) \in \mathrm{Cris}(X/S)$ and $\theta\colon p((U \hookrightarrow T, \gamma)) = q(\mathcal{F}, T')$ is an equality in Sch_S. This means $T = T'$.

Definition 0.2. We *define a crystal in quasi-coherent modules* as a cartesian section of $\mathrm{Qcoh} \times_{\mathrm{Sch}_S} \mathrm{Cris}(X/S) \xrightarrow{p_1} \mathrm{Cris}(X/S)$.

Unwinding definitions, this amounts to assigning each $(U \hookrightarrow T, \gamma) \in \mathrm{Cris}(X/S)$ an object $(\mathcal{F}, T)$ of Qcoh and each arrow $f\colon (U \hookrightarrow T, \gamma) \to (U' \hookrightarrow T', \gamma')$ an arrow $(f, \theta) : (\mathcal{F}, T) \to (\mathcal{F}', T')$ so that $\theta\colon f^*\mathcal{F}' \to \mathcal{F}$ is an isomorphism in $\mathrm{Qcoh}(T)$.

References

[Gro66] Lettre de A.Grothendieck á J.Tate, 1966. A transcribed copy of the original by M. Carmona is available at https://agrothendieck.github.io/divers/LGT66.pdf

[SP] The Stacks project, The Stacks project authors, https://stacks.math.columbia.edu, 2021